QUASI-METRIC CONNECTIONS AND A CONJECTURE OF CHERN ON AFFINE MANIFOLDS

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Our main result is:

Theorem 0.1. *The Euler characteristic of a closed even dimensional affine manifold is zero.*

Let $\xi = (E, \pi, M^n)$ be an oriented vector bundle over an n dimensional manifold M^n, and ∇ a connection in ξ. Here E denotes the total space of ξ and $\pi : E \mapsto M$is the canonical projection onto M. Let g be a positively defined metric on E. The usual meaning of the compatibility of a connection ∇ on E with the metric g is by requiring the metric g to be paralel with respect to the connection that is

$$\nabla g \equiv 0. \tag{1}$$

In order to prove our result we have to weaken this compatibility condition as follows

Definition 0.2. *We say that ∇ is quasi-compatible with the metric g if for every $p \in M$ there exist a local frame $(e_i)_{i=1,\cdots,n}$, orthonormal at p such that the matrix of connection forms with respect to $(e_i)_{i=1,\cdots,n}$ is skew-symmetric at p. Such a local frame will be called a compatible frame at p.*

1. The Euler form of a general linear connection

This section describes the construction of the Euler form of a general linear connection. For technical details we will also refer the reader to [1] and [2]. In what follows the manifold M is a smooth, closed and even dimensional manifold of dimension $n = 2m$. Let us briefly remember the construction of the Euler form associated to a Levi Civita connection. Let M be an n-dimensional oriented manifold, g a Riemannian metric, and D its associated Levi Civita connection. Let $(e_i)_{i=1,\cdots,n}$ be a positive local orthonormal frame with respect to g and let $(\theta_i)_{i=1,\cdots,n}$ be the connection forms with respect to the frame $(e_i)_{i=1,\cdots,n}$. They are defined by the equations

$$De_j = \theta_{ij} e_i. \tag{2}$$

The matrix (θ_{ij}) is skew-symmetric. The curvature forms are defined by Cartan's second structural equation

$$\Omega_{ij} = d\theta_{ij} - \theta_{ik} \wedge \theta_{kj} \tag{3}$$

and the matrix (Ω_{ij}) is skew symmetric as well. The matrix (Ω_{ij}) globally defines an endomorphism of the tangent bundle, and therefore the trace is independent of the choice of the local frame (e_i). Moreover, since the matrix (Ω_{ij}) is skew-symmetric, its determinant is a "perfect square", hence the square root is also invariant under a change of the positive local frame. A heuristic definition of the Euler form of D is

$$\mathcal{E}(D) = \sqrt{det\Omega}. \tag{4}$$

From (4) we see that $\mathcal{E}$ is an n-form defined globally on M, hence it defines a cohomology class.

To be able to define the Euler form of a general linear connection we need first some linear algebra. Let V be a $n = 2m$-dimensional vector space and let A be a skew-symmetric matrix with 2-forms as entries, that is

$$A \in \Lambda^2(V, so(2m, \mathbb{R})).$$

The Pfaffian Pf is map

$$Pf : \Lambda^2(V, so(2m, \mathbb{R})) \mapsto \Lambda^{2m}(V).$$

which, for a matrix

$$A = \begin{bmatrix} 0 & a_{1,2} & .. & a_{1,2m} \\ -a_{1,2} & 0 & .. & a_{2,2m} \\ .. & .. & .. & .. \\ -a_{2m,1} & -a_{2m,2} & .. & 0 \end{bmatrix},$$

is defined as

$$\mathrm{Pf}(A) = \sum_{\alpha \in \Pi} sgn(\alpha) a_\alpha. \tag{5}$$

Here $a_\alpha = a_{i_1,j_1} \wedge a_{i_2,j_2} \wedge .. \wedge a_{i,j_m}$ and Π is the set of all partitions of the set $\{1, 2, 3, ...2m\}$ into pairs of elements. Since every element α of Π can be represented as

$$\alpha = \{(i_1, j_1), (i_2, j_2), .., (i_m, j_m)\},$$

and since any permutation π associated to α has the same signature as

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & .. & 2m \\ i_1 & j_1 & i_2 & j_2 & .. & j_m \end{bmatrix},$$

the equality (5) makes sense. The following lemma will allow us to define the Euler form of a general linear connection.

Lemma 1.1. *Let $A, B \in \Lambda^2(V, M_n(2m, \mathbb{R}))$ be two orthogonally equivalent matrices , that is*

$$B = U^T A U,$$

for some orthogonal matrix U with positive determinant. Then

$$Pf(A - A^T) = Pf(B - B^T)$$

Proof. Since

$$B = U^T A U, \tag{6}$$

it follows that

$$B^T = (U^T A U)^T = U^T A^T U \tag{7}$$

Substracting (7) from (6) we obtain

$$B - B^T = U^T (A - A^T) U,$$

which shows that $B - B^T$ and $A - A^T$ are skew-symmetric and orthogonally equivalent. The conclusion of the lemma follows. □

Lemma (1.1) will allow us to construct the Euler form of a general linear connection. However in order to define the Euler form we will need a reference positive definite metric on the vector bundle.

Let $E \to M$ be a vector bundle endowed with a positive definite metric g. Let ∇ be a connection on E(not necesarily compatible to g). Let $p \in M$ and $(\sigma_i)_{i=1,\cdots,n}$ be an orthonormal frame at p. Let Ω be the matrix of the curvature forms of the connection ∇ at p with respect to the frame $(\sigma_i)_{i=1,\cdots,n}$. The matrix Ω is not, in general, skew-symmetric! However we can define the Pfaffian for $\frac{\Omega - \Omega^T}{2}$ and by virtue of Lemma (1.1) we obtain a global form on M.

Definition 1.2. *The Euler form of a connection ∇ on a bundle E endowed with a positive definite metric g is defined as*

$$\mathcal{E}_g(\nabla) = (2\pi)^{-n} Pf\left(\frac{\Omega - \Omega^T}{2}\right) \tag{8}$$

Next we will prove that the Euler form, defined as in (8) is a closed form for any ∇ quasi-compatible with the metric g. We need the following linear algebra lemma. The proof of the lemma can be found in [2](see pages 296-297).

Lemma 1.3. *Let $Pf(A)$ be the Pfaffian of a skew matrix $A = (a_{ij})$ and $P'(A) = (\frac{\partial P}{\partial a_{ji}})$ be the transpose of the matrix obtained by formally differentiating the Pfaffian with respect to the indetrminates a_{ij}. Then the following are true:*
(a) $P'(A)A = AP'(A)$
(b) If the entries of A are differential forms, then $dPf(A) = Tr(P'(A)dA)$.

Before we prove that the Euler form of a connection, quasi-compatible with a metric is closed in any dimension, let us consider the case of a surface Σ endowed with a Riemannian metric g. Take ∇ a connection quasi-compatible to g, and let $p \in \Sigma$. Take e_1, e_2 a compatible local frame at p and let ω_{ij} and Ω_{ij} be the connection and curvature matrices respectively. We would like to show that

$$dPf(\Omega - \Omega^T)(p) = 0. \tag{9}$$

The Bianchi identity states that

$$d\Omega_{ij} = \omega_{ik} \wedge \Omega_{kj} - \Omega_{ik} \wedge \omega_{kj} \tag{10}$$

Since the Pfaffian of a two by two matrix is just the bottom left corner of the matrix we have

$$dPf(\Omega - \Omega^T) = d\Omega_{21} - d\Omega_{12} \tag{11}$$

$$= \omega_{2k} \wedge \Omega_{k1} - \Omega_{2k} \wedge \omega_{k1} - \omega_{1k} \wedge \Omega_{k2} + \Omega_{1k} \wedge \omega_{k2}$$

Specializing (11) at p, where ω_{ij} is skew, and expanding we get

$$dPf(\Omega - \Omega^T)(p) = \omega_{21} \wedge \Omega_{11} - \Omega_{22} \wedge \omega_{21} - \omega_{12} \wedge \Omega_{22} + \Omega_{11} \wedge \omega_{12} \tag{12}$$

$$= \omega_{21} \wedge \Omega_{11} - \omega_{21} \wedge \Omega_{11} + \omega_{21} \wedge \Omega_{22} - \omega_{21} \wedge \Omega_{22} = 0.$$

The equation (9) shows that the Euler form (as defined in this paper by equation (8)) is closed for any connection in $T\Sigma$ that is quasi-compatible with a metric g.

Next we will prove that the Euler form is closed in the general case. Let $\xi = (E, \pi, M^n)$ be a vector bundle, g a positive definite metric and ∇ a connection quasi-compatible with g on E. We begin by showing that $\Omega - \Omega^T$ satisfies a Bianchi identity at p. That is

$$d(\Omega - \Omega^T)(p) = \omega \wedge (\Omega - \Omega^T) - (\Omega - \Omega^T) \wedge \omega. \tag{13}$$

To see this, let us consider e_i a local compatible frame for ∇ at p. We have

$$d\Omega_{ij} = \omega_{ik} \wedge \Omega_{kj} - \Omega_{ik} \wedge \omega_{kj} \tag{14}$$

and

$$d\Omega_{ji} = \omega_{jk} \wedge \Omega_{ki} - \Omega_{jk} \wedge \omega_{ki}, \tag{15}$$

and substracting (15) from(14) we get

$$\begin{aligned} d(\Omega - \Omega^T) = \omega_{ik} \wedge \Omega_{kj} - \Omega_{ik} \wedge \omega_{kj} - \\ -\omega_{jk} \wedge \Omega_{ki} + \Omega_{jk} \wedge \omega_{ki} \end{aligned} \tag{16}$$

which specialized at p where ω_{ij} are skew we get

$$d(\Omega - \Omega^T)(p) = \omega_{ik} \wedge (\Omega_{kj} - \Omega_{jk}) - (\Omega_{ik} - \Omega_{ki}) \wedge \omega_{jk} \tag{17}$$

which in matrix notation is

$$d(\Omega - \Omega^T)(p) = \omega \wedge (\Omega - \Omega^T) - (\Omega - \Omega^T) \wedge \omega \tag{18}$$

Next, according to Lemma 1.3 part (b) we have

$$dPf(\Omega - \Omega^T)(p) = Tr(P'(\Omega - \Omega^T) d(\Omega - \Omega^T)) \tag{19}$$

which, at p, combined with equation (18) gives

$$dPf(\Omega - \Omega^T)(p) = Tr(P'(\Omega - \Omega^T)(\omega \wedge (\Omega - \Omega^T) - (\Omega - \Omega^T) \wedge \omega). \tag{20}$$

Using Lemma 1.3 part (a) we get

$$\begin{aligned} dPf(\Omega - \Omega^T)(p) = \\ Tr(P'(\Omega - \Omega^T)\omega \wedge (\Omega - \Omega^T) - (\Omega - \Omega^T) \wedge P'(\Omega - \Omega^T)\omega) \end{aligned} \tag{21}$$

Setting

$$A = P'(\Omega - \Omega^T)\omega$$

equation 21 becomes

$$dPf(\Omega - \Omega^T)(p) = \sum (A_{ij} \wedge (\Omega - \Omega^T)_{ji} - (\Omega - \Omega^T)_{ji} \wedge A_{ij}), \tag{22}$$

which because of commutativity of wedge products with 2 forms gives

$$dPf(\Omega - \Omega^T)(p) = 0. \tag{23}$$

Summing up we get

Theorem 1.4. *The Euler form of a connection ∇, quasi-compatible with a positively defined metric g is a closed form. It therefore defines a cohomology class of M.*

2. Proof of main result

We wil now prove Theorem 0.1

Proof.

Let g be a global Riemannian metric on M that has D as its Levi Civita connection. First we will prove that ∇ (the affine connection on M) can be continuously deformed into the global metric connection D through g quasi-compatible connections. Using this homotopy we will prove that $\mathcal{E}$, the Euler form of ∇, and $\mathcal{E}'$ the Euler form of D, represent the same cohomology class.
We begin by constructing a one parameter family of g-quasi-compatible connections on TM denoted ∇^t for $t \in [0,1]$. Take $p \in M$. Let U_p be a contractible affine neighborhood of p. Since the restricted holonomy group of ∇ with respect to p is trivial, then there exist a UNIQUE Riemannain metric h^p on U_p such that

$$\nabla h^p \equiv 0$$

and

$$h^p(p) = g(p). \tag{24}$$

Consider the metric on U_p defined by

$$h^{t,p} = (1-t)h^p + tg \tag{25}$$

and let $D^{t,p}$ be its Levi Civita connection. Let X be a tangent vector field on U_p and $v \in T_pM$. We define the covariant derivative of a vector field at p

$$(\nabla^t)_p : T_pM \times \mathcal{X}(U) \to T_pM \tag{26}$$

as

$$\nabla^t_v X = D^{t,p}_v X. \tag{27}$$

From its construction it is obvious that

$$(\nabla^t h^{t,p})(p) = 0 \tag{28}$$

and that

$$\nabla^0 = \nabla \tag{29}$$

and

$$\nabla^1 = D. \tag{30}$$

Moreover we have the identity

$$h^{t,p}(p) = (1-t)h^p(p) + tg(p) = (1-t)g(p) + tg(p) = g(p). \tag{31}$$

Now let us prove that the connection forms of ∇^t with respect to any $h^{t,p}$-orthonormal frame are skew-symmetric. Let $(e_i)_{i=1,\cdots,n}$ a local orthonormal frame with respect to $h^{t,p}$ on the open neighborhood U_p. Let X be a vector field on U_p. We have

$$X(h^{t,p}(e_i, e_j)) = h^{t,p}(D^{t,p}_X e_i, e_j) + h^{t,p}(e_i, D^{t,p}_X e_j). \tag{32}$$

Since

$$h^{t,p}(e_i, e_j) = \delta_{ij},$$

on U_p, it follows that

$$0 = h^{t,p}(D^{t,p}_X e_i, e_j) + h^{t,p}(e_i, D^{t,p}_X e_j). \tag{33}$$

Taking into account equation (27) and if we denote by ω_{ij} the connection forms of the connection ∇^t and since at p the metric $h^{t,p}$ coincides with g(see equation (31)) it follows that

$$0 = \omega_{ik}\delta_{kj} + \omega_{jk}\delta_{ki} = \omega_{ij} + \omega_{ji}. \tag{34}$$

This proves the quasicompatibility of ∇^t with g. According to Theorem 1.4 it follows that its Euler form is closed.

Now let $\pi : M \times [0,1] \to M$ be defined as

$$\pi(p,t) = p.$$

First we need to prove that the deformation ∇^t of ∇ into D defines a quasi-metric connection on $\tau = \pi^*(TM)$. We set

$$\pi^*(\nabla^t) = D_t.$$

On τ we also have the pullback metric from M which we denote g^*. We define the connection $\mathbb{D}$ on τ by defining its action on a smooth section σ of τ

$$(\mathbb{D}\sigma)(x,t) = (D_t\sigma)(x,t). \tag{35}$$

From its definition it's obvious that its connection forms with respect to the pullback of a local $h^{t,p}$-orthonormal frame from M are just the pulback of the connection forms of ∇^t from M. Hence $\mathbb{D}$ is quasi-metric

and its Euler form $\mathcal{A}$ is well defined . Also according to Theorem 1.4 we have that

$$d\mathcal{A} = 0.$$

We define a family of maps

$$i_t : M \to M \times [0, 1]$$

by

$$i_t(x) = (x, t).$$

Since the Euler form behaves nicely with respect to pullbacks (see [1] the proof of Lemma 18.2), we have

$$i_0^* \mathcal{A} = \mathcal{E}$$

and

$$i_1^* \mathcal{A} = \mathcal{E}'.$$

Because the two maps i_0 and i_1 are homotopic and $\mathcal{A}$ is closed, they induce the same map in cohomology and it follows that

$$\mathcal{E} - \mathcal{E}'$$

is exact on M, and the conclusion of the theorem follows. □

References

[1] Madsen M. and Tornehave J. , From calculus to cohomology: de Rham cohomology and characteristic classes, Cambridge University Press, Cambridge 1997.

[2] Milnor J. and Stasheff J., Characteristic Classes, Annals of Mathematics Studies, Princeton University Press 1974.